Simple Solutions for Hyperbolic and Related Position Fixes

BERTRAND T. FANG
The Analytic Sciences Corp.

Navigation fixed from range differences to three stations plus an additional piece of information are investigated. It is shown that if the additional information is the navigator altitude, or the range difference to a fourth station, the computation of the navigation fix is reduced to finding the roots of a quadratic. If the additional information is the range to another station, or that the navigator is on the Earth ellipsoid, the fix can be obtained by solving a quartic. By emphasizing the underlying geometric interpretations, these fixes and their simple solutions are made clear. The derivations also show that the same solution algorithms are applicable if the basic navigation measurements are range sums instead of range differences.

Manuscript received October 18, 1989; revised December 18, 1989.

IEEE Log No. 38444.

Author's address: The Analytic Sciences Corp., 12100 Sunset Hills Rd., Reston, VA 22090.

0018-9251/90/0900-0748 $1.00

INTRODUCTION

Navigation systems such as LORAN or DECCA [1] use differences in the times of arrival of a radio signal at different stations to determine a navigation position. It is well known that time of arrival differences at a pair of stations locate the navigator on a hyperboloid of revolution with foci at the stations; that time arrival differences at three stations place the navigator ont he curve of intersection of two such hyperboloids. To fix the position at a point on this curve of intersection requires additional information. Some examples of such information are: the position is on the surface of an ellipsoidal Earth or another station exists which provides additional signal time of arrival differences. Navigation positions located in this way at the intersections of hyperboloids and other surfaces may be called hyperbolic position fixes. Usually, computing a hyperbolic position fix requires an iterative algorithm with its attendant inefficiency and convergence problem [1]. It is shown in the following that the computation of these hyperbolic position fixes can be reduced to the solution of a quadratic or a quartic equation. The simplicity of these solutions comes from the use of station baseline planes as references and from exploring the geometrical properties of intersection of hyperboloids. The advantages of such references was first noted in a related navigation problem [2].

Measurements which are sums of signal times of arrival are also common. These measurements involve ellipsoids and lead to the elliptic position fixes. It is obvious from our derivation below that the algorithms derived for hyperbolic position fixes are also applicable to elliptic position fixes.

NAVIGATION POSITION RELATIVE TO THREE STATIONS

Fig. 1 shows a navigation position relative to three stations A, B, and C. A set of local right-handed orthogonal axes is chosen as shown. The origin is at one of the stations, one axis is along a station baseline, and another axis is orthogonal to the two station baselines, or the station plane.

Let V be the signal velocity, $T_{ab} = T_a - T_b$ and $T_{ac} = T_a - T_c$ be the differences in the times of signal arrival at the station pairs A, B and A, C, respectively. From Fig. 1, one has

$$\sqrt{x^2+y^2+z^2} - \sqrt{(x-b)^2+y^2+z^2} = V * T_{ab} = R_{ab} \tag{1}$$

$$\sqrt{x^2+y^2+z^2} - \sqrt{(x-c_x)^2+(y-c_y)^2+z^2} = V * T_{ac} = R_{ac} \tag{2}$$

where R_{ab} and R_{ac} are range differences from the navigation position to the stations, converted from

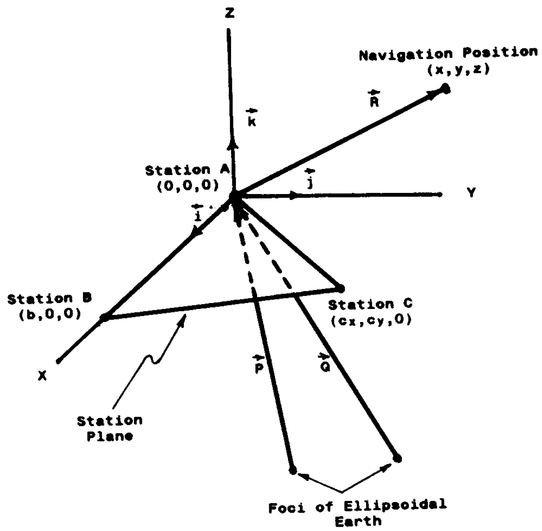


Fig. 1. Navigation position referenced to local coordinates defined by station baselines.

the measured time of arrival differences. Transposing the first terms to the right-hand sides of (1) and (2), squaring and simplifying, one obtains

$$R_{ab}^2 - b^2 + 2b*x = 2R_{ab}\sqrt{x^2+y^2+z^2} \quad (3)$$

$$R_{ac}^2 - c^2 + 2c_x*x + 2c_y*y = 2R_{ac}\sqrt{x^2+y^2+z^2} \quad (4)$$

where b and $c = \sqrt{c_x^2 + c_y^2}$ are the lengths of station baselines. These two equations, when squared, can be readily recognized as representing two hyperboloids of revolution with foci at A, B and A, C, respectively.

Note that for measurements consisting of sums instead of differences of times of arrival, one would have a set of equations similar to (1) and (2) with positive signs between the radicals and with R_{ab} and R_{ac} interpreted as range sums instead of range differences. Squaring these new equations would give a set of equations identical to (3) and (4), although representing ellipsoids of revolution rather than hyperboloids. Thus all derivations below which originate from (3) and (4) are applicable to range sum measurements as well as range difference measurements.

Let us consider, without loss of generality, that the range difference R_{ab} is not equal to zero.[1] Then equating (3) and (4) and simplifying, one obtains

$$y = g*x + h \quad (5)$$

where

$$g = \{R_{ac}*(b/R_{ab}) - c_x\}/c_y \quad (6)$$

$$h = \{c^2 - R_{ac}^2 + R_{ac}*R_{ab}(1-(b/R_{ab})^2)\}/2c_y. \quad (7)$$

Substituting (5) into (3), one obtains

$$z = \pm\sqrt{d*x^2 + e*x + f} \quad (8)$$

or

$$z^2 = d*x^2 + e*x + f \quad (9)$$

where

$$d = -\{1 - (b/R_{ab})^2 + g^2\} \quad (10)$$

$$e = b*\{1-(b/R_{ab})^2\} - 2g*h \quad (11)$$

$$f = (R_{ab}^2/4)*\{1-(b/R_{ab})^2\}^2 - h^2. \quad (12)$$

These equations admit the following geometric interpretations. Equation (5) defines a plane orthogonal to the station baselines. The navigation position must lie in this plane, or the curve of intersection of the two hyperboloids is a plane curve. Equation (8) says this curve must be symmetrical with respect to the station plane. Equation (9) says the projection of this curve onto the $X-Z$ plane is an ellipse ($d < 0$) or a hyperbola ($d > 0$). A simple expression for this curve of intersection can be obtained by a straightforward transformation of coordinate axes such that the new origin is on the plane defined by (5) and the new Y-axis is orthogonal to the plane. We do not go into the details but will point out that, as its projection discussed above, this curve is an ellipse or a hyperbola depending on whether $d < 0$ or $d > 0$. From (10) it can be seen that for range sum measurements, $d < 0$ and the curve is an ellipse, being the intersection of a plane and an ellipsoid of revolution. For range difference measurements this curve is the intersection of a plane and a hyperboloid of revolution, and may be a hyperbola, or an ellipse. It can be seen from (6) and (10) that an ellipse would result if the angle subtended by the two baselines is small.

One may now write the navigation position vector as follows which depends on a single unknown parameter x,

$$\vec{R} = x*\vec{i} + (g*x+h)\vec{j} \pm \sqrt{d*x^2+e*x+f}\vec{k}. \quad (13)$$

As discussed above, this vector defines a hyperbola ($d > 0$) or an ellipse ($d < 0$) with mirror symmetry with respect to the station plane.

POSITION FIX WITH ADDITIONAL INFORMATION

The preceding section shows that when time of arrival differences or sums to three stations are known, an ellipse or a hyperbola on which the navigator lies can be computed. To fix the navigation position on this

[1] $R_{ab} = 0$ implies $x = b/2$. In that case one can follow a similar procedure and express (13) below in terms of y instead of x. But a separate treatment is unnecessary because one can choose a baseline such that $R_{ab} = 0$, unless $R_{ab} = R_{ac} = 0$. In the latter situation the problem is trivial since the navigation position will be at an equal distance to the three stations.

ellipse or hyperbola, additional information is required. Among the commonly available information, some will restrict the position on another plane; others will place it on a second degree surface. For the former, the navigation position becomes the solution of a quadratic equation. This is easy to understand from geometry. The intersection of this new plane with the plane of the hyperbola or ellipse is a straight line. And the intersection of this first degree straight line with a coplanar second degree curve such as a hyperbola or an ellipse is a root of a quadratic equation. For the latter, the intersection of the plane of the hyperbola or ellipse with a second degree surface is a second degree planar curve. Thus the navigation position is at the intersection of two coplanar second degree curves, or the root of a quartic equation.

Expressions for these quadratic and quartic equations are derived below. Before proceeding, however, it is to be noted that in the derivations that led to (13) we removed radicals by squaring appropriate expressions. This process can introduce extraneous solutions. Only those roots which satisfy the measurement equations (1) and (2) are admissible navigation solutions. That extraneous roots may exist can be seen from (6), (7), (10)–(12) that the parameters d, e, f, g, h and therefore the vector R as given in (13) remain unchanged if R_{ab} and R_{ac} are replaced by $-R_{ab}$ and $-R_{ac}$.

A. Altitude of Navigator Above Station Plane Known

An example of this situation is the local (flat Earth approximation) navigation of an aircraft equipped with an altimeter. Since the altitude z is known, x is obtainable as the solution of (9), i.e.,

$$d*x^2 + e*x + (f - z^2) = 0. \tag{9a}$$

Geometrically the navigation position is at the intersection of the plane z =known altitude with a hyperbola or an ellipse and where two admissible solutions corresponding to the two roots of (9a) generally exist. This two-fold ambiguity can often be resolved if some knowledge of the general location of the navigator is available.

B. Signal Arrival Time Difference or Sum to Another Station Known

This is the problem of the hyperbolic or elliptic position fix; i.e., the navigation position is at the intersection of three hyperboloids or ellipsoids. For this case, consider another station C′ and the associated timing measurement T'_{ac} are available. The stations A, B, C′ provide another set of reference and the timing measurements T'_{ac} and T_{ab} define another plane on which the navigator lies. An alternative expression for the navigation position vector referenced to the stations A, B and C′ is, similar to (13),

$$\vec{R} = x*\vec{i} + (g'*x + h')*\vec{j}' \pm \sqrt{d'x^2 + e'*x + f'}\vec{k}' \tag{14}$$

where the primed quantities are computed just like the corresponding unprimed quantities, with the station C replaced by the station C′. Taking the scalar product of (13) and (14) with the unit vector $\vec{j}'$ and equating the results, one obtains

$$g'*x + h' = (g*x + h)(\vec{j}*\vec{j}') \pm \sqrt{d*x^2 + e*x + f}*(\vec{k}*\vec{j}')$$

or, squaring and simplifying

$$p*x^2 + q*x + r = 0 \tag{15}$$

where

$$p = d*(\vec{k}*\vec{j}')^2 - \{g' - g*(\vec{j}*\vec{j}')\}^2 \tag{16}$$

$$q = e*(\vec{k}*\vec{j}') - 2\{g' - g*(\vec{j}*\vec{j}')\}\{h' - h*(\vec{j}*\vec{j}')\} \tag{17}$$

$$r = f*(\vec{k}*\vec{j}')^2 - \{h' - h*(\vec{j}*\vec{j}')\}^2. \tag{18}$$

With x known as the solution of (15), y and z follow from (5) and (8), respectively. As discussed already, for the present situation, the navigation position is at the intersection of a straight line and a hyperbola or an ellipse, and generally has two solutions. If the fourth station C′ is not in the plane of the stations A, B and C, these two solutions correspond to two values of x which are the two roots of (15). The $\pm$ sign of z in (8) can be resolved, because symmetries with respect to the A, B, C and A, B, C′ planes are incompatible. On the other hand, if the four stations are coplanar, the two planes containing the navigation position and defined by the two sets of references must intersect at a line parallel to the Z-axis, (15) must have double roots and the two possible navigation positions are mirror images with respect to the station plane corresponding to the $\pm$ signs of z in (8).

C. Navigator on Ellipsoid of Revolution

To a very good approximation, the surface of the ocean is an ellipsoid of revolution. Thus this is the situation for the navigation of ships. Since a sphere is a special case of an ellipsoid, this also includes the special case that the range of the navigator to another location is known, the location of interest may be another station or the center of the Earth. As discussed already, the navigation position is now at the intersection of an ellipse with a coplanar hyperbola or another ellipse, and it is obvious from geometry that two or four points of intersection may exist. The quartic that governs these intersections can be derived as follows. Let the position vectors from the two foci of the ellipsoid to Station A be $\vec{P}$ and $\vec{Q}$, respectively (Fig. 1). From the defining property of an ellipsoid of revolution, one must have

$$\sqrt{(\vec{P}+\vec{R})*(\vec{P}+\vec{R})}+\sqrt{(\vec{Q}+\vec{R})*(\vec{Q}+\vec{R})}=2a \quad (19)$$

where a is the semimajor axis of the ellipsoid. Transposing and squaring twice, one obtains

$$\{(P^2-Q^2-4a^2)+2(\vec{P}-\vec{Q})*\vec{R}\}^2$$
$$=16a^2*(Q^2+2\vec{Q}*\vec{R}+R^2)$$

where P, Q, and R are the lengths of the vectors $\vec{P}$, $\vec{Q}$, and $\vec{R}$, respectively. By expressing the known vectors $\vec{P}$ and $\vec{Q}$ in terms of their components along the $X-Y-Z$ axes defined in Fig. 1, it can be readily seen that the above equation can be rearranged as follows

$$u*x^2+v*x+w=\pm\sqrt{d*x^2+e*x+f}*(n*x+m) \quad (20)$$

where

$$u=\{(\text{beta})^2+d*s_z^2\}/4a^2-(1+g^2+d)$$
$$v=\{(\text{alpha})*(\text{beta})+e*s_z^2\}/4a^2$$
$$-(2q_x+2g*q_y+e+2g*h)$$
$$w=\{(\text{alpha})^2+4f*s_z^2\}/16a^2$$
$$-(Q^2+2h*q_y+h^2+f)$$
$$m=-s_z*(\text{alpha})/4a^2+2q_z$$
$$n=-s_z*(\text{beta})/2a^2$$
$$\text{alpha}=P^2-Q^2-4a^2+2s_y*h$$
$$\text{beta}=s_x+g*s_y$$

q_x, q_y, q_z and s_x, s_y, s_z are components of vector $\vec{Q}$ and $\vec{S}=\vec{P}-\vec{Q}$ along X, Y, and Z axes, respectively.

Notice that if the ellipsoid becomes a sphere, i.e., for the special case that the range of the navigator to a known location is given, then $n=0$, $m=2q_z$, the first terms in the expressions for u and v vanish, that for w becomes a^2, and the algebra simplifies considerably. Squaring (20), one obtains the following quartic for x

$$x^4(u^2-d*n^2)+x^3(2u*v-e*n^2-2d*m*n)$$
$$+x^2(v^2+2u*w-f*n^2-d*m^2-2e*m*n)$$
$$+x(2v*w-2f*m*n-e*m^2)+(w^2-f*m^2)=0 \quad (21)$$

When x is known, y and z follow from (5) and (8). Note that the quartic may have four distinct real roots. These, together with the $\pm$ values of z means there are eight possible combinations. However, as discussed before, only two or four of these are admissible navigation positions. The others are extraneous solutions which do not satisfy the measurement equations (1) and (2), or the auxiliary information (19).

INDEPENDENT BASELINES

Sometimes different station baselines are independent, i.e., the station clocks are synchronized only in pairs. In that case, although two sets of time of arrival differences, say T_{ab} and T_{cd} still constrain the navigation position at the intersection of two hyperboloids, the curve of intersection is no longer a planar curve, and the simple results obtained in previous sections no longer apply. However, as long as there is a set of three synchronized stations, additional measurements from independent baselines do provide simple solutions. Obviously, if an additional set of three synchronized stations exists, the navigation fix is again given by the roots of a quadratic. Likewise, if an additional independent baseline exists, the navigation fix is given by the roots of a quartic. The derivations parallel those in B and C above and are not reported here.

COMPUTATION FLOW

To illustrate the solution algorithm, the computational flow for a navigation fix on the Earth ellipsoid is presented in Fig. 2. The computations for other fixes are similar, but simpler, particularly when the solution is governed by a quadratic instead of a quartic.

DISCUSSION

It is shown in the above that several problems of interest in computing hyperbolic (elliptic) navigation fixes can be reduced to the solution of a quadratic or quartic equation. The solution of a quadratic is trivial. Analytic solution of a quartic is available, although some algebra is involved, but it is a simple matter to program the algorithm on a computer, as has been done by the author. The simplicity of the solutions results from the recognition that the intersection of two hyperboloids (ellipsoids) of revolution with a common focus is a hyperbola or an ellipse symmetrical with respect to the plane of the foci. By exploring the geometrical interpretations the nature of the navigation fixes are clarified. When the navigation position is governed by the quadratic, generally two admissible navigation positions exist and the ambiguity must be resolved from other information such as knowledge of the general whereabouts of the navigator. The quartic may have four distinct real roots corresponding to four possible navigation positions. Frequently some are extraneous roots which can be rejected by showing that

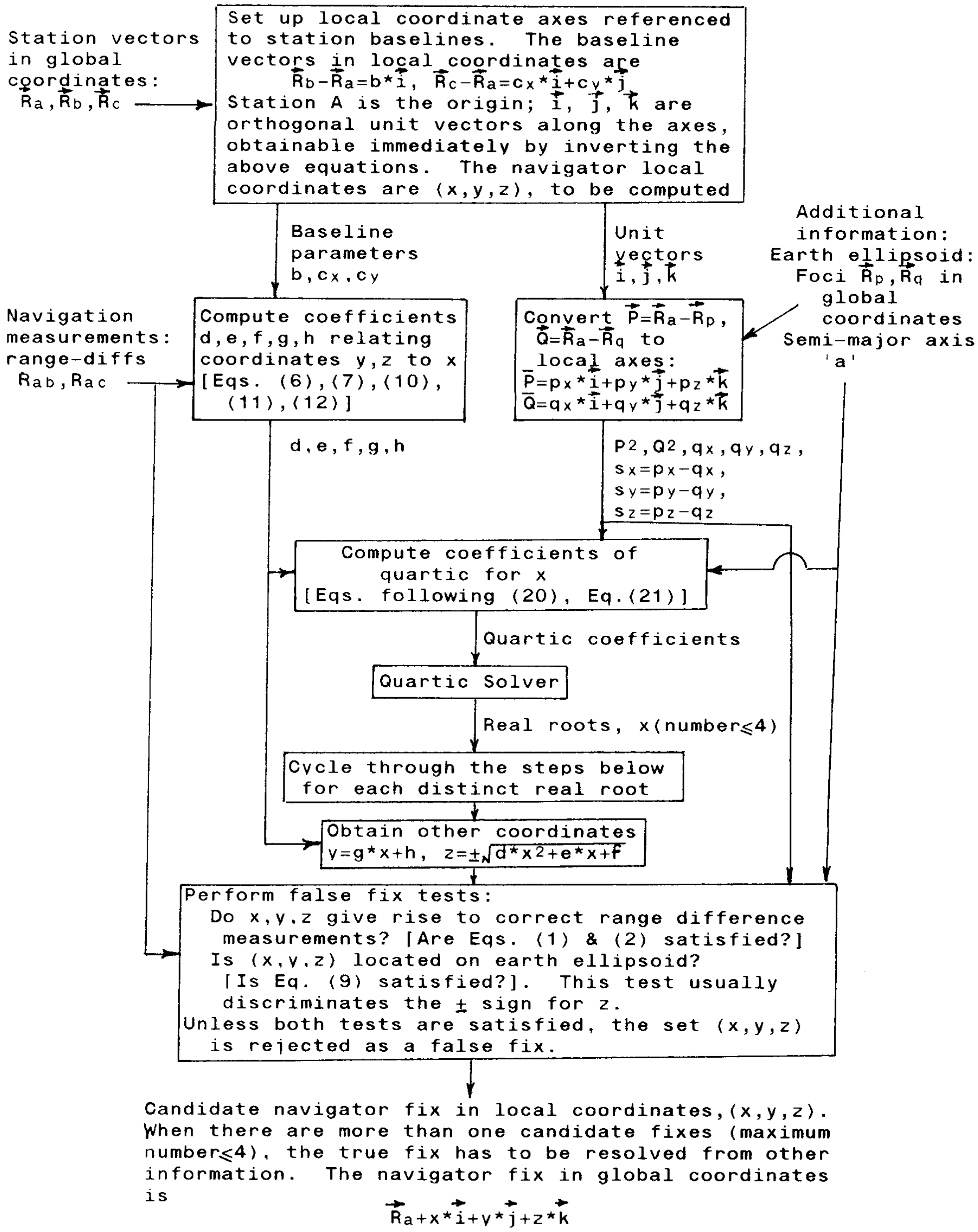


Fig. 2. Computation flow.

they do not produce the correct measurements. If a navigation position exists, it must be one of the real roots of the quadratic or the quartic, as the case may be. The nonexistence of an admissible root indicates a gross measurement error.

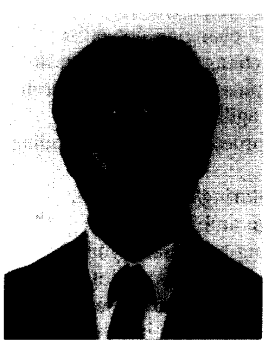
The navigation fixes discussed are based on range difference or range sum information, converted from time difference and time sum measurements. For terrestrial navigation systems that rely on ground wave propagation such conversion can be complicated. For line-of-sight wave propagation, or navigation in space, the conversion is straightforward. Very long baseline interferometry (time difference) and bilateration using a remote ground transponder (time-sum) are examples of such space navigation systems [3].

REFERENCES

[1] Kayton, M., and Freid, W. R. (1969)
Avionics Navigation Systems.
New York: Wiley, 1969.

[2] Fang, B. T. (1986)
Trilateration and extension to global positioning system navigation.
Journal of Guidance, Control, and Dynamics, **9** (Nov.–Dec. 1986), 715–717.

[3] Ray, J., Knight, C., Zelensky, N., Liebrecht, P., and Jones, E. (1988)
VLBI tracking of the TDRS.
Journal of Astronautical Sciences, **36** (Oct.–Dec. 1988), 347–364.



Bertrand T. Fang was born in China on February 2, 1932. He received the B.S. degree in mechanical engineering from National Taiwan University, Taiwan, in 1952, the M.S. degree in theoretical and applied mechanics from Iowa State University, Ames, in 1957, and the Ph.D. in aeronautical engineering from University of Minnesota, Minneapolis, in 1962.

Dr. Fang taught aerospace engineering at the Catholic University of America in Washington, DC for eleven years. His industrial experience includes employment at the General Electric Company, E G & G Washington Analytic Service Center, Inc., and Computer Sciences Corporation. Since 1985 he has been with The Analytic Sciences Corporation.